# ASYMPTOTIC FORM OF THE PROBLEM OF NONSYMMETRICAL PROPER VIBRATIONS OF A CIRCULAR-CONICAL SHELL

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The problem concerning nonsymmetric proper vibrations of a circular-conical shell with a vertex is separated into two parts, one being the determination of an edge-effect type of solution, and the other being the determination of a more slowly varying solution. The ranges of wavelengths for which the separation in question becomes asymptotically accurate as the thickness of the shell tends to zero are indicated. The first solution is written down explicitly; the second is obtained for a free and for a swivel-supported boundary by numerical methods. The numerical solution indicates that the system of equations governing the slowly varying part of the solution has a winding point of a rather complicated structure.

1. We describe the proper vibrations of a circular-conical shell made of a homogeneous isotropic elastic material by V. Z. Vlasov's [1, pp. 253, 368] system of two differential equations for the stress function v(x, y, t) and the deformation function w(x, y, t) ($x \in [0, a]$ and $y \in [0, 2\pi]$ are the meridional and angular coordinates on the middle surface and t is the time). We represent the well-known expressions for the longitudinal stresses $N_1$, S, $N_2$ and the flexural deformations $\varkappa_1$, $\tau$, $\varkappa_2$ in terms of the functions v, w in the form

$$N_1 = -BR^{-2}(l^2D_2^2 + q_1D_1)v, \quad S = BR^{-2}l(D_1 - q_1)D_2v,$$
$$N_2 = -BR^{-2}D_1^2v$$
$$\varkappa_2 = -R^{-2}(l^2D_2^2 + q_1D_1)\, w, \; \tau = -R^{-2}l(D_1 - q_1)D_2w, \varkappa_1 = -R^{-2}D_1^2w$$
$$(B = CEh, \quad C = h/\sqrt{12(1-\nu^2)})$$
$$(l = l(x) = a/x, \quad q_1 = q_1(x) = 1/x, \quad a = 1/\sin\theta). \tag{1.1}$$

Here E is the elastic modulus, $\nu$ is the coefficient of transverse expansion of the material, h is the constant thickness, R is the radius of the base of the shell, $D_1$ and $D_2$ are operators denoting single differentiations with respect to the coordinates x and y, respectively, and $\theta$ is the angle that the generators of the cone make with the axis of rotation.

At the vertex (x = 0) of the shell we require that the static and geometrical characteristics of the deformation, determined by expressions (1.1), be finite, and at its boundary ($x = a$) we prescribe the following boundary conditions ($M_1$ is the meridional bending moment and $Q_1$ is the meridional transverse stress):

$$N_1 = 0, \; S = 0, \; M_1 = 0, \; Q_1 = 0 \text{ or } w = 0 \tag{1.2}$$

One of these two systems of conditions determines a free boundary (the case $Q_1 = 0$), while the other (w = 0) determines a variant of the case of a swivel-supported boundary.

The problem consists of the search for special solutions of the original system, having the form

$$v_n(x, y, t) = \varphi_n(x)\psi_n(y, t), \quad w_n(x, y, t) = \chi_n(x)\psi_n(y, t),$$
$$\psi_n(y, t) = \exp(iny + i\omega t) \tag{1.3}$$

where n is an integer, $\omega$ is a real number, and $i = \sqrt{-1}$.

Insertion of expressions (1.3) into V. Z. Vlasov's equations results in the following system of two ordinary differential equations for the functions $\varphi_n(x)$ and $\chi_n(x)$:

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$$P_\mu \varphi_n + 2lD_1^2\chi_n = 0, \quad 2lD_1^2\varphi_n - (P_\mu - \lambda)\chi_n = 0$$
$$(P_\mu = \mu^2 P_2^{(4)} - 2\mu\mu_n l^2 P_1^{(2)} + \mu_n^2 l^4, \quad P_2^{(4)} = Q^{(2)}Q^{(2)}, \quad P_1^{(2)} = 0.5\,(Q^{(2)} + l^{-2}Q^{(2)}l^2)$$
$$Q^{(2)} = D_1^2 + q_1 D_1, \quad \lambda = 2\omega\rho R^2 / \mu^2 E \cos\theta, \quad \mu^2 = 2C / R\cos\theta, \quad \mu_n = \mu n^2) \tag{1.4}$$

where $\rho$ is the density of the material, and indices in parentheses indicate the order of the differential operator.

Equations (1.4) have a regular singularity at zero. Only those of their solutions that are analytic at $x=0$ have physical meaning. From (1.2) we have for the boundary conditions at the point $x=a$

$$\varphi_n = 0, \quad D_1\varphi_n = 0, \quad G_1^{(2)}\chi_n = 0, \quad \gamma\chi_n - (1-\gamma)\,G_2^{(3)}\chi_n = 0$$
$$(G_1^{(2)} = D_1^2 + \nu q_1 D_1, \quad G_2^{(3)} = D_1(\mu P^{(2)} - \mu_n l^2) - \mu_n l^2 (1-\nu)\,(D_1 - q_1)) \tag{1.5}$$

Here an auxiliary parameter, assuming the values 0 and 1, has been introduced. For the cases when $\gamma = 0$ we have $Q_1 = 0$, i.e., a free boundary, and in the other case we have $w = 0$, i.e., a swivel-supported boundary.

Solution of the problem on vibrations of the shell under consideration involves the determination of those positive eigenvalues $\lambda_m$ ($m = 1, 2, 3, \ldots$) of the system (1.4) which correspond to the eigenfunctions $\varphi_{nm}$ and $\chi_{nm}$, satisfying the required conditions.

2. We restrict the range of values of the geometrical parameter $\mu$ by the condition $\mu^2 \ll 1$, thereby excluding shells with very mild slopes from consideration. At the same time the quantity $\mu_n^2$ is not necessarily small, for the number n can be arbitrarily large. We assume [2, 3] that

$$n = C_0 \mu^{-\alpha} \qquad (\alpha \geqslant 0, \ C_0 \approx 1) \tag{2.1}$$

The system (1.4) consists of equations with a small parameter multiplying the highest-order derivatives. For $\mu = 0$ it determines a certain degenerate system, which does not in general coincide with the system of the moment-free theory. For values $\alpha > 1/2$ the degenerate system is trivial. Setting this case aside for the present, we restrict values of the parameter $\alpha$ in (2.1) by the condition

$$0 \leqslant \alpha \leqslant 1/2 \tag{2.2}$$

In this domain we determine a system of equations that is shortened as compared to (1.4)

$$P^{(2)}\varphi_n^\circ + 2lD_1^2\chi_n^\circ = 0, \quad 2lD_1^2\varphi_n^\circ - (P^{(2)} - \lambda^\circ)\chi_n^\circ = 0$$
$$(P^{(2)} = -2\mu\mu_n l^2 P_1^{(2)} + \mu_n^2 l^4) \tag{2.3}$$

which differs from the degenerate case only in terms that vanish for $\mu \to 0$. However, thanks to these terms, it preserves the same singularity, and consequently the same character of the solution at zero as is possessed by the full system (1.4).

The shortened system (2.3) is of fourth order. This is the same order as that of the system of the moment-free theory, which follows from (2.3) when $P^{(2)} \equiv 0$. In the transition from (1.4) to (2.3) we dropped the highest-order derivatives with small parameters — those terms which generate solutions of the edge-effect type.

From results of [4] it follows that at a sufficient distance from the point $x = a$ the solution of the shortened system determines the solution of the full system within an error of order not exceeding $\mu^2$. In the neighborhood of the point $x = a$, however, the difference may turn out to be substantial in view of the fact that the solution of the shortened system does not have a sufficient degree of arbitrariness for the satisfaction of all the boundary conditions. Discrepancies that arise can be compensated by solutions of the edge-effect type.

To develop such solutions we "freeze" the coefficients of Eqs. (1.4) at some point $x = \xi$ $(0 < \xi \leq a)$ (see [5]). As a result, we have a system of equations with constant coefficients

$$P_{\mu\xi}\varphi_n + 2l_\xi D_1^2\chi_n = 0, \quad 2l_\xi D_1^2\varphi_n - (P_{\mu\xi} - \lambda)\chi_n' = 0$$

Introducing a function $z_n(x)$, which can be differentiated a sufficient number of times and is such that

$$\varphi_n = -2l_\xi D_1^2 z_n, \quad \chi_n = P_{\mu\xi} z_n \tag{2.4}$$

we reduce this system to one eight-order differential equation

$$Q_\xi(D_1)z_n = \lambda\mu_n^2 l^4 z_n$$

$$Q_\xi(D_1) = \mu^4 Q_{4\xi}^{(8)} - 4\mu^3\mu_n l_\xi^2 Q_{3\xi}^{(6)} + (4l_\xi^2 + 6\mu^2\mu_n^2 l_\xi^4 - \lambda\mu^2) Q_{2\xi}^{(4)}$$

$$- 2\mu\mu_n l_\xi^2 (2\mu_n^2 l_\xi^4 - \lambda) Q_{1\xi}^{(2)} + \mu_n^4 l_\xi^8, \quad Q_{i\xi}^{(k)} = D_1^k + \sum_{j=0}^{k-1} q_{ij}(\xi) D_1^j, \tag{2.5}$$

it being possible that certain of the coefficients $q_{ij}$ depend on positive powers of the parameter $\mu$.

We require that the condition

$$q_2 \equiv 4l^2 + 6\mu^2\mu_n^2 l^4 - \lambda\mu^2 \geqslant O(1) \tag{2.6}$$

holds over the entire segment $0 \le x \le a$.

Then the degenerate operator $Q_\xi(D_1)$ will be of fourth order for any $\xi$ in the interval $0 < \xi \le a$. We ensure that this condition will be fulfilled by making the following restriction on the value of the parameter $\lambda$ [6]:

$$\lambda = O(\mu^{-\beta}), \quad \beta < 2 \tag{2.7}$$

The characteristic equation $Q_\xi(s) = 0$ has eight roots:

$$s_k(\xi, \lambda, \mu) \text{ and } s_{k+4}(\xi, \lambda, \mu) = \mu^{-1} r_k(\xi, \lambda, \mu) \quad (k = 1, 2, 3, 4),$$

$s_k$ and $r_k$ being analytic functions of the parameter $\mu$ (see Lemma 1 of [5]). With an error not exceeding the order of $\mu$, the values $r_k$ are determined from the algebraic equation

$$r_k^4 + q_2(\xi) = 0$$

By virtue of (2.6) we have $|\mathrm{Re}(\{r_k\})| \geq O(1)$, so that (for fixed $\xi$) this equation determines four pairwise conjugate complex numbers. Two of them have negative, and two of them have positive real parts. Therefore the degeneration of the operator $Q_\xi(D_1)$ into an operator of fourth order is regular (in the sense of [5]) for any $0 < \xi \le a$.

To each of the values $r_k$ there corresponds the particular solution

$$z_{k\xi}(x) = \exp(\mu^{-1} r_k(\xi) x)$$

of Eq. (2.5). In terms of it we also determine a particular solution of the original system (1.4) in the neighborhood of the point $x = \xi$ with the aid of Eq. (2.4). Let $r_1$ and $r_2$ have positive, and $r_3$ and $r_4$ have negative real parts. With $\xi = a$ solutions that decrease with distance from the boundary correspond to the roots $\mu^{-1}r_1$ and $\mu^{-1}r_2$, while those that increase correspond to $\mu^{-1}r_3$ and $\mu^{-1}r_4$. Consequently, only the first two solutions have the character of an edge effect. Therefore the general solution of the edge-effect type has the form

$$z_{na} = \sum_{k=1}^{2} C_k \exp(\mu^{-1} r_k(a)(x - a))$$

in the neighborhood of the point $x = a$.

The functions that correspond to this

$$\varphi_n^I = -2D_1^2 z_{na}, \quad \chi_n^I = P_{\mu a} z_{na} \tag{2.8}$$

describe the rapidly-varying part of the solution of system (1.4).

According to a theorem on asymptotic representation [4] we have

$$\lambda_m = \lambda_m^\circ + \mu\delta_m, \quad \varphi_{nm} = \varphi_{nm}^\circ + \varphi_{nm}^I + \mu\zeta_{nm}, \quad \chi_{nm} = \chi_{nm}^\circ + \chi_{nm}^I + \mu\eta_{nm} \tag{2.9}$$

if only the functions $\varphi_{nm}^o + \varphi_{nm}^I$ and $\chi_{nm}^o + \chi_{nm}^I$ satisfy the boundary conditions (1.5) ($\lambda_m^o$ is a simple eigenvalue of the shortened system, $\varphi_{nm}^o$ and $\chi_{nm}^o$ are eigenfunctions, $\varphi_{nm}^I$ and $\chi_{nm}^I$ are edge-effect types of solutions answering to the value $\lambda = \lambda_m^o$, and, in order of magnitude, the norm of the quantities $\delta_m$, $\xi_{nm}$, $\eta_{nm}$ does not exceed the terms of the other term standing ahead of them in the corresponding expressions). There is the possibility of satisfying all the boundary conditions, as these functions contain four arbitrary constants: the solution of the system (2.3), subjected to the condition of analyticity at zero, gives two of them, and the solution of (2.8) gives the other two.

We shall not attempt to separate the four conditions (1.5) into two pairs of such a character that, when one pair is assigned to the shortened system (2.3) and the other to the equation (2.5) for the edge effects, the solutions of the equations can be determined sequentially without violating the asymptotic representation (2.9).

To separate out those conditions to which the general solution of the edge-effect type must be subjected we make a substitution of (2.4) into (1.5). Then the boundary conditions at the point $x=a$ assume the form

$$A_1 z_n = 0,\ A_2 z_n = 0,\ A_3 z_n = 0,\ A_4(\gamma) z_n = 0$$
$$(\gamma = 0, 1)$$
$$(A_1 = D_1^2,\ A_2 = D_1^3,\ A_3 = G_1^{(2)} P_{\mu a}^{(4)},\ A_4(\gamma) = [\gamma - (1-\gamma) G_2^{(3)}] P_{\mu a}^{(4)}) \tag{2.10}$$

As the coefficient of the last two boundary operators depend on the small parameter $\mu$, we use a modification [3] of the well-known rule for separating boundary conditions [4, 7].

If the boundary conditions have the form

$$A_i^{(k)} z \equiv \sum_{j=0}^{k} a_{ij} D_1^j z = 0 \qquad (i = 1, 2, \ldots, l) \tag{2.11}$$

with $a_{ij} = O(\mu^{\alpha_{ij}})$, then, first of all, they must not contain a small parameter raised to a negative power, and this parameter must not be a common factor of some of the conditions. In other words, Eq. (2.11) must be replaced by

$$\mu^{-\alpha_i} A_i^{(k)} z = 0, \qquad \alpha_i = \inf_j (\alpha_{ij}) \tag{2.12}$$

Then the characteristic exponent $\beta_{ij}$ of each term of (2.12) must be determined by the formula

$$\beta_{ij} = j + \alpha_i - \alpha_{ij}$$

The boundary conditions have a canonical form if each of them is solved for the term with the largest characteristic exponent $\beta_i = \sup_j (\beta_{ij})$ and if all of them are arranged in the order of strict increase in their largest exponents. After the canonical form of the boundary conditions has been established, their separation is carried out according to the rule: we assign to the equation for the edge effects the same number of boundary conditions as it has particular solutions of the edge-effect type at a given boundary point. The remaining boundary conditions are assigned to the equations governing the slowly varying part of the solution.

We turn immediately to condition (2.10). Assuming that

$$\nu q_1(a) = \nu \sin\theta = O(\mu^{\alpha_0})$$

we determine their largest characteristic exponents

$$\beta_1 = 2,\ \beta_2 = 3,\ \beta_3 = \inf_{(\alpha, \alpha_0)} (6 - 2\alpha,\ 6 - 4\alpha + \alpha_0),\ \beta_4 = \begin{cases} 7 - 6\alpha & \text{for } \gamma = 0 \\ 4 - 4\alpha & \text{for } \gamma = 1 \end{cases}$$

For the values of $\alpha$ in (1.2) we have

$$\beta_3 \geqslant 4,\ \beta_4 \geqslant \{4 \text{ for } \gamma = 0,\ 2 \text{ for } \gamma = 1\}$$

Since, in the case $\gamma = 0$ (free boundary) the characteristic exponents $\beta_3$ and $\beta_4$ are always larger than $\beta_1$ and $\beta_2$, the last two of conditions (1.5) must be assigned to Eq. (2.5) or to its solution (2.8) at the point $x=a$, while the first two are assigned to the system (2.3). In the case $\gamma = 1$ (swivel-supported boundary) matters are somewhat more complicated. For $0 \leq \alpha < 1/4$ the boundary conditions separate in the same way as they do for the case $\gamma = 0$, but for $1/4 < \alpha < 1/2$ the characteristic exponents $\beta_2$ and $\beta_3$ are larger than $\beta_1$ and $\beta_4$ so that we must assign the second and third of conditions (1.5) to (2.5), while the first and fourth are assigned to (2.3).

Now we can give a final formulation of separation problems. In writing down the boundary conditions for the shortened system we replace $\varphi_n$ with $\varphi_n^\circ$ and $\chi_n$ with $\chi_n^\circ$ in the corresponding conditions (1.5). This means that at the point $x=a$

$$\varphi_n^\circ = 0,\ D_1 \varphi_n^\circ = 0 \text{ for } \gamma = 0,\ 0 \leqslant \alpha < 1/2 \text{ and for } \gamma = 1,\ 0 \leqslant \alpha < 1/4 \tag{2.13}$$

$$\varphi_n^\circ = 0,\ \chi_n^\circ = 0 \text{ for } \gamma = 1,\ 1/4 < \alpha < 1/2 \tag{2.14}$$

(as before, we require analyticity of the functions $\varphi_n^\circ$ and $\chi_n^\circ$ at zero). We obtain the boundary conditions for the edge-effect type of solution (2.8) from the corresponding conditions (1.5) by replacing $\varphi_n$ with $\varphi_n^\circ + \varphi_n^I$ and $\chi_n$ with $\chi_n^\circ + \chi_n^I$. As the result we have at the point $x=a$

$$G_1^{(2)}\chi_n{}^I = -G_1^{(2)}\chi_n{}^\circ,\ G_2^{(3)}\chi_n{}^I = -G_2^{(3)}\chi_n{}^\circ \begin{cases} \text{for} \quad \gamma = 0,\ 0 \leqslant \alpha < {}^1/{}_2 \text{ and} \\ \text{for} \quad \gamma = 1,\ 0 \leqslant \alpha < {}^1/{}_4 \end{cases} \tag{2.15}$$

$$D_1\varphi_n{}^I = -D_1\varphi_n{}^\circ,\ G_1^{(2)}\chi_n{}^I = -G_1^{(2)}\chi_n{}^\circ \text{ for } \quad \gamma = 1,\ {}^1/{}_4 < \alpha < {}^1/{}_2 \tag{2.16}$$

From the asymptotic representation (2.9) it follows that when conditions (1.5) are satisfied the values $\lambda_m{}^0$ and the functions $\varphi^0_{mn} + \varphi^I_{mn}$, $\chi^0_{nm} + \chi^I_{nm}$ determine the eigenvalues and the eigenfunctions of the full problem within an error not exceeding the order $\mu$. If it is required only that this error vanish together with $\mu$, that is, if one intends only that the solution have the asymptotic behavior of that of the full problem, then the domain of values of the parameter $\alpha$ can be widened to the range $0 \le \alpha < 1$. In all of this region the characteristic equation $Q_\xi(s) = 0$ has four roots of the form $\mu^{-1} r_k$ and four roots of lower order. At the point $x = a$ two particular solutions of the edge-effect type correspond to the first roots. Consequently the conditions, making it possible to separate the original system of equations, are fulfilled. The question concerning the separation of the boundary conditions is resolved, as above, by determining their characteristic exponents. As we now have

$$\beta_3 > 2,\ \beta_4 > (1 \text{ for } \quad \gamma = 0,\ 0 \text{ for } \quad \gamma = 1)$$

any of conditions (2.10), which means any of conditions (1.5), can appear as boundary conditions for the system (2.3).

3. The existence of solutions of the system (1.4) that not only diminish rapidly, but also grow rapidly with distance from the boundary, causes serious difficulties in the numerical integration of the system. Difficulties of this kind arise in many boundary problems of shell theory. In some papers special methods are applied to cope with them. Thus, to this end, the method of suppression of rapidly increasing solutions is applied in [8], in [9] a modification of the pivotal condensation method is used, while [10] employs the method of orthogonalization. All these methods enable one to widen the domain of permissible values of the parameter $\mu$ to a greater or lesser degree; nevertheless they lose their effectiveness for sufficiently small values of the parameter.

In contrast to these, the asymptotic method becomes more effective for smaller values of $\mu$. It separates out in explicit form the rapidly-varying solutions that are unfavorable from the point of view of difference approximations, and the application of numerical methods to the solution of the shortened (degenerate) problem causes no difficulty as its solutions are smoother. Moreover, the order of the shortened system of equations is two less than that of the original system.

As an illustration of the statements made above concerning the character of the solution of the shortened problem we give the results of a numerical integration of the system (2.3) under the conditions (2.13) and (2.14) at the point $x = a$ and the conditions $\varphi_n{}^0 = \chi_n{}^0$ at the point $x = 0$. The results were obtained by the pivotal condensation method [11]. As an object for the calculation we took a conical shell with the geometric parameters: $\theta = \pi/6$ (i.e., $a = 2$), $\mu^2 = 1/200$.

The eigenvalues of the frequency parameter $\lambda^0$, calculated to third-place accuracy (which required no more than 50 coordinate steps), are given in Table 1.

The first row gives values of the waveform parameter n for which calculations were performed. The second and third rows give the first eigenvalues of the shortened problem with the boundary conditions (2.13) and (2.14) respectively. The fourth row gives the second eigenvalues, corresponding to the boundary conditions (2.13). All the eigenvalues given satisfy the condition $\beta < 2$. As n varies from 2 to 9, the parameter $\alpha$ varies within the range $0.25 \leqslant \alpha \leqslant 0.65$. For a shell with a free boundary ($\gamma = 0$) the boundary conditions, corresponding to the shortened system, have the form (2.13) throughout this range. Therefore, all the numbers given in the second row can be understood as approximations to the eigenvalues of the full problem $\gamma = 0$. For a shell with a swivel-supported boundary ($\gamma = 1$) the boundary conditions have the form (2.14) in the domain $0.25 < \alpha \leqslant 0.65$. For $\alpha \approx 0.25$ the shortened problem does not give the asymptotic behavior of the full problem either for conditions (2.13) or for (2.14). The value $n = 2$ corresponds to this value of $\alpha$. Consequently, in the case $\gamma = 1$, the approximate eigenvalues of the full problem are the numbers in the third row, which answer to values $n \geq 3$.

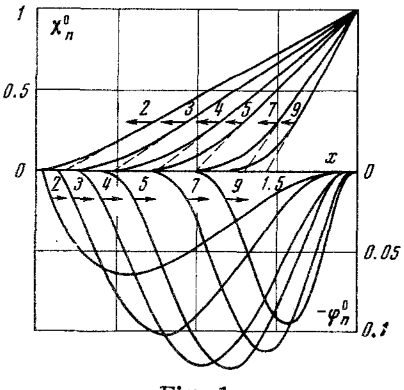


Fig. 1

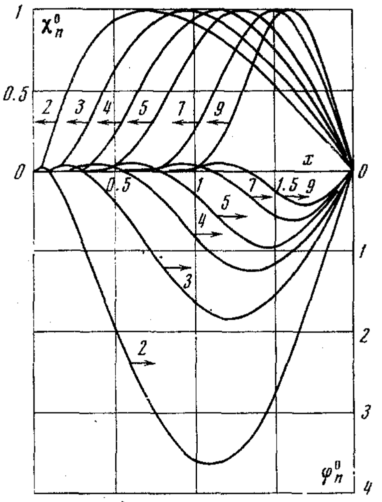

Fig. 2

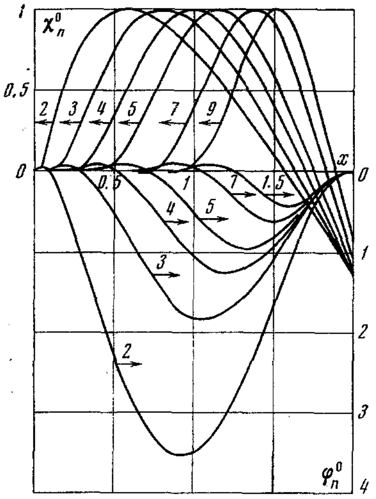

Fig. 3

TABLE 1

| $n$ | 2 | 3 | 4 | 5 | 7 | 9 |
|---|---|---|---|---|---|---|
| $\lambda_1^\circ$ | 0.22 | 0.66 | 1.50 | 2.94 | 8.70 | 20.6 |
| $\lambda_1^\circ$ | 23.3 | 15.5 | 15.3 | 18.3 | 30.9 | 55.0 |
| $\lambda_2^\circ$ | 37.8 | 24.5 | 23.3 | 26.3 | 40.7 | 67.8 |

Figures 1–3 give a graphical representation of the eigenfunctions corresponding to the eigenvalues in Table 1. The curves of Fig. 1 correspond to values in the second row, those of Fig. 2 to values in the third row, and those of Fig. 3 to values in the fourth row. The side of the figure on which the ordinates of the various curves are to be read off is indicated by the direction of the arrows attached to the curves, while the value of the parameter n to which a curve corresponds is indicated by the number over the arrow.

A characteristic property of all the curves represented is the occurrence of inflection points, that is, points on either side of which the asymptotic representation of the functions $\varphi_n^0$ and $\chi_n^0$ is different. For example, as one moves to the left of the inflection point, the functions $\chi_n^0$, represented in Fig. 1, diminish exponentially, while to the right of the inflection point they increase linearly with x. The inflection points themselves are approximately determined by the intersection of the dashed lines [sic] with the axis of abscissae. The inflection points of the functions $\varphi_n^0$ in Fig. 1 and the functions $\varphi_n^0$, $\chi_n^0$ in Fig. 2 separate branches of exponential form from branches of oscillatory form. With increasing values of the parameter n the inflection points of both functions are displaced toward the boundary $x = a$, the solution becoming increasingly localized in the neighborhood of the boundary.

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